MATH 2050A: Mathematical Analysis I (2016 1st term)

1 Sequentially Compact Sets and Compact Sets in $\mathbb{R}$

Definition 1.1 Let A be a subset of $\mathbb{R}$. A point $z \in \mathbb{R}$ is called a limit point of A if for any $\delta > 0$, there is an element $a \in A$ such that $0 < |z - a| < \delta$.
Put $D(A)$ the set of all limit points of A.

Example 1.2 (i) $D([a, b]) = D((a, b)) = [a, b]$.

(ii) $D([0, 1] \cup \{2\}) = [0, 1]$.

(iii) $D(\mathbb{N}) = \emptyset$.

(iv) $D(\{a\}) = \emptyset$ for any $a \in \mathbb{R}$.

Definition 1.3 A subset A of $\mathbb{R}$ is said to be closed in $\mathbb{R}$ if $D(A) \subseteq A$.

Example 1.4 (i) $\{a\}; [a, b]; [0, 1] \cup \{2\}$; $\mathbb{N}$ and $\mathbb{R}$ all are closed subsets of $\mathbb{R}$.

(ii) (a, b) and $\mathbb{Q}$ are not closed.

The following Lemma can be directly shown by the definition, so, the proof is omitted here.

Lemma 1.5 *Let A be a subset of $\mathbb{R}$. The following statements are equivalent.*

(i) A is closed.

(ii) For each element $x \in \mathbb{R} \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$.

(iii) If (x_n) is a sequence in A and $\lim x_n$ exists, then $\lim x_n \in A$.

Definition 1.6 Let A be a subset of $\mathbb{R}$.

(i) A is said to be sequentially compact if every sequence (x_n) in A has a convergent subsequence (x_{n_k}) with $\lim_k x_{n_k} \in A$.

(ii) A is said to be compact if for any open intervals cover $\{J_\alpha\}_{\alpha \in \Lambda}$ of A, that is, each J_α is an open interval and

$$A \subseteq \bigcup_{\alpha \in \Lambda} J_\alpha,$$

we can find finitely many $J_{\alpha_1}, .., J_{\alpha_N}$ such that $A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$.

Example 1.7 (i) Every closed and bounded interval is sequentially compact.
In fact, if (x_n) is any sequence in a closed and bounded interval $[a,b]$, then (x_n) is bounded. Then by Bolzano-Weierstrass Theorem (see [1, Theorem 3.4.8]), (x_n) has a convergent subsequence (x_{n_k}). Notice that since $a \leq x_{n_k} \leq b$ for all k, then $a \leq \lim_k x_{n_k} \leq b$, and thus $\lim_k x_{n_k} \in [a,b]$. Therefore A is sequentially compact.

(ii) $(0,1]$ is not sequentially compact. In fact, if we consider $x_n = 1/n$, then (x_n) is a sequence in $(0,1]$ but it has no convergent subsequence with the limit sitting in $(0,1]$.

(iii) $(0,1]$ is not compact. In fact, if we put $J_n = (1/n, 2)$ for $n = 2,3...$, then $(0,1] \subseteq \bigcup_{n=2}^{\infty} J_n$, but we cannot find finitely many $J_{n_1},...,J_{n_K}$ such that $(0,1] \subseteq J_{n_1} \cup \cdots \cup J_{n_K}$. So $(0,1]$ is not compact.

Theorem 1.8 (Heine-Borel Theorem) *Every closed and bounded interval* $[a,b]$ *is a compact set.*

Proof: Suppose that $[a,b]$ is not compact. Then there is an open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of $[a,b]$ but it it has no finite sub-cover. Let $I_1 := [a_1,b_1] = [a,b]$ and m_1 the mid-point of $[a_1,b_1]$. Then by the assumption, $[a_1,m_1]$ or $[m_1,b_1]$ cannot be covered by finitely many J_α's. We may assume that $[a_1,m_1]$ cannot be covered by finitely many J_α's. Put $I_2 := [a_2,b_2] = [a_1,m_1]$. To repeat the same steps, we can obtain a sequence of closed and bounded intervals $I_n = [a_n,b_n]$ with the following properties:

(a) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots$;

(b) $\lim_n(b_n - a_n) = 0$;

(c) each I_n cannot be covered by finitely many J_α's.

Then by the Nested Intervals Theorem (see [1, Theorem 2.5.2, Theorem 2.5.3]), there is an element $\xi \in \bigcap_n I_n$ such that $\lim_n a_n = \lim_n b_n = \xi$. In particular, we have $a = a_1 \leq \xi \leq b_1 = b$. So, there is $\alpha_0 \in \Lambda$ such that $\xi \in J_{\alpha_0}$. Since J_{α_0} is open, there is $\varepsilon > 0$ such that $(\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. On the other hand, there is $N \in \mathbb{N}$ such that a_N and b_N in $(\xi - \varepsilon, \xi + \varepsilon)$ because $\lim_n a_n = \lim_n b_n = \xi$. Thus we have $I_N = [a_N, b_N] \subseteq (\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. It contradicts to the Property (c) above. The proof is finished. □

Theorem 1.9 *Let* A *be a subset of* $\mathbb{R}$. *The following statements are equivalent.*

(i) A *is compact.*

(ii) A *is sequentially compact.*

(iii) A *is closed and bounded.*

Proof: The result is shown by the following path $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.
For $(i) \Rightarrow (ii)$, *suppose that* A *is compact but it is not sequentially compact. Then there is a sequence* (x_n) *in* A *such that* (x_n) *has no subsequent which has the limit in* A. *Put* $X = \{x_n : n = 1,2,...\}$. *Then* X *is infinite. Also, for each element* $a \in A$, *there is* $\delta_a > 0$ *such that* $J_a := (a - \delta_a, a + \delta_a) \cap X$ *is finite. Indeed, if there is an element* $a \in A$ *such that* $(a - \delta, a + \delta) \cap A$ *is infinite for all* $\delta > 0$, *then* (x_n) *has a convergent subsequence with the*

limit a. On the other hand, we have $A \subseteq \bigcup_{a\in A} J_a$. *Then by the compactness of A, we can find finitely many* $a_1, ..., a_N$ *such that* $A \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. *So we have* $X \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. *Then by the choice of* J_a*'s, X must be finite. This leads to a contradiction. Therefore, A is sequentially compact.*

For (ii) ⇒ (iii), assume A is sequentially compact. We first claim that A must be bounded. Otherwise, if A is unbounded and if we fix $x_1 \in A$, *then there is an element* $x_2 \in A$ *such that* $|x_1 - x_2| > 1$. *By the unboundedness of A again, we can find* $x_3 \in A$ *such that* $|x_3 - x_k| > 1$ *for* $k = 1, 2$. *To repeat the same step, we can obtain a sequence* (x_n) *in A such that* $|x_m - x_n| > 1$ *for* $m \neq n$. *Thus* (x_n) *has no convergent subsequence and hence A is not sequentially compact. So A must be bounded if A is sequentially compact.*

Secondly, we will see that A must be closed. Let (x_n) *be a sequence in A such that* $a := \lim x_n$ *in* $\mathbb{R}$. *We want to show that* $a \in A$ *by Lemma 1.5. In fact, since A is sequentially compact, then* (x_n) *has a convergent subsequence* (x_{n_k}) *with* $\lim_k x_{n_k} \in A$. *This gives* $a = \lim_n x_n = \lim_k x_{n_k} \in A$. *So, A is closed. Part (ii) follows.*

It remains to show (iii) ⇒ (i). Suppose that A is closed and bounded. Then we can find a closed and bounded interval $[a, b]$ *such that* $A \subseteq [a, b]$. *Now let* $\{J_\alpha\}_{\alpha\in\Lambda}$ *be an open intervals cover of A. Notice that for each element* $x \in [a, b] \setminus A$, *there is* $\delta_x > 0$ *such that* $(x - \delta_x, x + \delta_x) \cap A = \emptyset$ *since A is closed. If we put* $I_x = (x - \delta_x, x + \delta_x)$ *for* $x \in [a, b] \setminus A$, *then we have*

$$[a, b] \subseteq \bigcup_{\alpha\in\Lambda} J_\alpha \cup \bigcup_{x\in[a,b]\setminus A} I_x.$$

Using the Heine-Borel Theorem 1.8, we can find finitely many J_α*'s and* I_x*'s, say* $J_{\alpha_1}, ..., J_{\alpha_N}$ *and* $I_{x_1}, ..., I_{x_K}$, *such that* $A \subseteq [a, b] \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N} \cup I_{x_1} \cup \cdots \cup I_{x_K}$. *Note that* $I_x \cap A = \emptyset$ *for each* $x \in [a, b] \setminus A$ *by the choice of* I_x. *Therefore, we have* $A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$ *and hence A is compact.*

The proof is finished. □

2 Complete subsets in $\mathbb{R}$

Definition 2.1 A sequence (x_n) in $\mathbb{R}$ is called a Cauchy sequence if for every $\varepsilon > 0$, there is a positive integer N such that $|x_m - x_n| < \varepsilon$ for all $m, n \geq N$.

Remark 2.2 A sequence (x_n) is not a Cauchy sequence if and only if there is $\varepsilon > 0$ such that for any positive integer N, we can find some positive integers m, n with $m, n \geq N$ satisfying $|x_m - x_n| \geq \varepsilon$.

Example 2.3 For each positive integer n, if we put $x_n = \sum_{k=1}^{n} 1/k$, then (x_n) is not a Cauchy sequence. Indeed, notice that for any positive integer n, we have

$$|x_{2n} - x_n| = \frac{1}{n+1} + \cdots + \frac{1}{2n} \geq \frac{n}{2n} = \frac{1}{2}.$$

So, if we take $\varepsilon = 1/2$, then for any positive integer N, we have $|x_{2N} - x_N| \geq \varepsilon$. Thus (x_n) is not a Cauchy sequence.

Proposition 2.4 *Every convergent sequence is a Cauchy sequence.*

Proof: Let (x_n) be a convergent sequence and $L = \lim x_n$. Then for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|L - x_n| < \varepsilon$ for all $n \geq N$. So, for any $m, n \geq N$, we have $|x_m - x_n| \leq |x_n - L| + |L - x_m| < \varepsilon$. Hence (x_n) is Cauchy. □

Definition 2.5 A subset A of $\mathbb{R}$ is said to be complete if for any Cauchy sequence (x_n) in A is convergent in A, that is, $x_n \in A$ for all n and $\lim x_n$ belongs to A.

The following result is one of important theorems in history.

Theorem 2.6 *$\mathbb{R}$ is complete, that is, every Cauchy sequence in $\mathbb{R}$ is convergent. Consequently, a sequence is convergent in $\mathbb{R}$ if and only if it is a Cauchy sequence.*

Proof: Let (x_n) be a Cauchy sequence in $\mathbb{R}$. We first claim that (x_n) must be bounded. Indeed, by the definition of a Cauchy sequence, if we consider $\varepsilon = 1$, then there is a positive integer N such that $|x_m - x_N| < 1$ for all $m \geq N$ and thus we have $|x_m| < 1 + |x_N|$ for all $m \geq N$. So, if we let $M = \max(|x_1|, ..., |x_{N-1}|, |x_N| + 1)$, then we have $|x_n| \leq M$ for all n. Hence (x_n) is bounded.
So, we can now apply the Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence (x_{n_k}). Let $L := \lim_k x_{n_k}$. We are going to show that $L = \lim_n x_n$.
Let $\varepsilon > 0$. Since (x_n) is Cauchy, there is $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for all $m, n \geq N$. On the other hand, since $\lim_k x_{n_k} = L$, we can find a positive integer K so that $|L - x_{n_k}| < \varepsilon$ for all $k \geq K$. Now if we choose $r \geq K$ such that $n_r \geq N$, then for any $n \geq N$, we have $|x_n - L| \leq |x_n - x_{n_r}| + |x_{n_r} - L| < 2\varepsilon$. Thus (x_n) is convergent with $\lim_n x_n = L$.
The finial assertion follows from Proposition 2.4 at once.
The proof is finished. □

Corollary 2.7 *Let A be a subset of $\mathbb{R}$. Then A is complete if and only if A is closed in $\mathbb{R}$*

Proof: Suppose that A is complete. Let (x_n) be a convergent sequence in A. Then it must be a Cauchy sequence by Proposition 2.4. By the definition of completeness, $\lim x_n \in A$ and thus A is closed.
Conversely, assume that A is closed in $\mathbb{R}$. Let (x_n) be a Cauchy sequence in A. Theorem 2.6 tells us that $\lim_n x_n$ exists. Since A is closed, $\lim_n x_n \in A$. The proof is finished. □

Corollary 2.8 *Every compact subset of $\mathbb{R}$ is complete.*

Proof: It follows from Theorem 1.9 and Corollary 2.7 at once. □

3 Continuous functions defined on compact sets

Throughout this section, let A be a non-empty subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ a function defined on A.

Proposition 3.1 *Suppose that f is continuous on A. If A is compact, then there are points c and b in A such that*

$$f(c) = \max\{f(x) : x \in A\} \text{ and } f(b) = \min\{f(x) : x \in A\}.$$

Proof: By considering the function $-f$ on A, it needs to show that $f(c) = \max\{f(x) : x \in A\}$ for some $c \in A$.

Method I:

We first claim that f is bounded on A, that is, there is $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$. Suppose not. Then for each $n \in \mathbb{N}$, we can find $a_n \in A$ such that $|f(a_n)| > n$. Recall that A is compact if and only if it is closed and bounded (see Theorem 1.9). So, (a_n) is a bounded sequence in A. Then by the Bolzano-Weierstrass Theorem, there is a convergent subsequence (a_{n_k}) of (a_n). Put $a = \lim_k a_{n_k}$. Since A is closed and f is continuous, $a \in A$, from this, it follows that $f(a) = \lim_k f(a_{n_k})$. It is absurd because $n_k < |f(a_{n_k})| \to |f(a)|$ for all k and $n_k \to \infty$. So f must be bounded. So $L := \sup\{f(x) : x \in A\}$ must exist by the Axiom of Completeness.

It remains to show that there is a point $c \in A$ such that $f(c) = L$. In fact, by the definition of supremum, there is a sequence (x_n) in A such that $\lim_n f(x_n) = L$. Then by the Bolzano-Weierstrass Theorem again, there is a convergent subsequence (x_{n_k}) of (x_n) with $\lim_k x_{n_k} \in A$. If we put $c := \lim_k x_{n_k} \in A$, then $f(c) = \lim_k f(x_{n_k}) = L$ as desired. The proof is finished.

Method II:

We first claim that f is bounded above. Notice that for each $x \in A$, there is $\delta_x > 0$ such that $f(y) < f(x) + 1$ whenever $y \in A$ with $|x - y| < \delta_x$ since f is continuous on A. Now if we put $J_x := (x-\delta_x, x+\delta_x)$ for each $x \in A$, then $A \subseteq \bigcup_{x \in A} J_x$. So, by the compactness of A, we can find finitely many $x_1, ..., x_N$ in A such that $A \subseteq J_{x_1} \cup \cdots \cup J_{x_N}$ and it follows that for each $x \in A$, we have $f(x) < 1 + f(x_k)$ for some $k = 1, ..., N$. Now if we put $M := \max\{1+f(x_1), ..., 1+f(x_N)\}$, then f is bounded above by M on A.

Put $L := \sup\{f(x) : x \in A\}$. It remains to show that there is an element $c \in A$ such that $f(c) = L$. Suppose not. Notice that since $f(x) \leq L$ for all $x \in A$, we have $f(x) < L$ for all $x \in A$ under this assumption. Therefore, by the continuity of f, for each $x \in A$, there are $\varepsilon_x > 0$ and $\eta_x > 0$ such that $f(y) < f(x) + \varepsilon_x < L$ whenever $y \in A$ with $|y - x| < \delta_x$. Put $I_x := (x - \eta_x, x + \eta_x)$. Then $A \subseteq \bigcup_{x \in A} I_x$. By the compactness of A again, A can be covered by finitely many $I_{x_1}, ..., I_{x_N}$. If we let $L' := \max\{f(x_1)+\varepsilon_{x_1}, ..., f(x_N)+\varepsilon_{x_N}\}$, then $f(x) < L' < L$ for all $x \in A$. It contradicts to L being the least upper bound for the set $\{f(x) : x \in A\}$. The proof is complete. □

Definition 3.2 We say that a function f is *upper semi-continuous (resp. lower semi-continuous)* on A if for each element $z \in A$ and for any $\varepsilon > 0$, there is $\delta > 0$ such that $f(x) < f(z) + \varepsilon$ (resp. $f(z) - \varepsilon < f(x)$) whenever $x \in A$ with $|x - z| < \delta$.

Remark 3.3 (i) It is clear that a function is continuous if and only if it is upper semi-continuous and lower semi-continuous. However, an upper semi-continuous function need not be continuous. For example, define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } \ x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

(ii) From the **Method II** above, we see that if f is upper semi-continuous (resp. lower semi-continuous) on a compact set A, then the function f attains the supremum (resp. infimum) on A.

Proposition 3.4 *If $f : A \to \mathbb{R}$ is continuous and A is compact, then the image $f(A)$ is compact. Furthermore, if f is injective, then the inverse map $f^{-1} : f(A) \to A$ is also continuous.*

Proof: Recall the fact that a subset of $\mathbb{R}$ is closed if and only if it is closed and bounded (see Theorem 1.9). So, it needs to show that $f(A)$ is a closed and bounded set. We first notice that $f(A)$ is bounded by Proposition 3.1. It remains to show that $f(A)$ is a closed subset of $\mathbb{R}$. Let $y \in \overline{f(A)}$. Then there is a sequence (x_n) in A such that $\lim f(x_n) = y$. Then by Theorem 1.9 again, there is a convergent subsequence (x_{n_k}) of (x_n) such that $\lim_k x_{n_k} \in A$. Since f is continuous, it follows that $y = \lim_k f(x_{n_k}) = f(\lim_k x_{n_k}) \in f(A)$ and thus $f(A)$ is closed.
Concerning the last assertion, let $B = f(A)$ and $g = f^{-1} : B \to A$. Suppose that g is not continuous at some $b \in B$. Put $a = g(b) \in A$. Then there are $\eta > 0$ and a sequence (y_n) in B such that $\lim y_n = b$ but $|g(y_n) - g(b)| \geq \eta$ for all n. Let $x_n := g(y_n) \in A$. So, by the compactness of A, there is a convergent subsequence (x_{n_k}) of (x_n) such that $\lim_k x_{n_k} \in A$. Let $a' = \lim_k x_{n_k}$. Then we have $f(a') = \lim_k f(x_{n_k}) = \lim_k y_{n_k} = b$. On the other hand, since $|g(y_n) - g(b)| \geq \eta$ for all n, we see that

$$|x_{n_k} - a| = |g(y_{n_k}) - g(b)| \geq \eta > 0$$

for all k and hence $|a' - a| > 0$. This implies that $a \neq a'$ but $f(a') = b = f(a)$. It contradicts to f being injective.
The proof is finished. □

Remark 3.5 The assumption of the compactness in the last assertion of Proposition 3.4 is essential. For example, consider $A = [0, 1) \cup [2, 3]$ and define $f : A \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in [0,1) \\ x - 1 & \text{if } x \in [2,3]. \end{cases}$$

Then $f(A) = [0, 2]$ and f is a continuous bijection from A onto $[0, 2]$ but $f^{-1} : [0, 2] \to A$ is not continuous at $y = 1$.

Example 3.6 By Proposition 3.4, it is impossible to find a continuous surjection from $[0, 1]$ onto $(0, 1)$ since $[0, 1]$ is compact but $(0, 1)$ is not. Thus $[0, 1]$ is not homeomorphic to $(0, 1)$.

Definition 3.7 A function $f : A \to \mathbb{R}$ is said to be uniformly continuous on A if for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in A$ with $|x - y| < \delta$.

Remark 3.8 It is clear that if f is uniformly continuous on A, then it must be continuous on A. However, the converse does not hold. For example, consider the function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) := 1/x$. Then f is continuous on $(0, 1]$ but it is not uniformly continuous on $(0, 1]$. Notice that f is not uniformly continuous on A means that

there is $\varepsilon > 0$ such that for any $\delta > 0$, there are $x, y \in A$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

Notice that $1/x \to \infty$ as $x \to 0+$. So if we let $\varepsilon = 1$, then for any $\delta > 0$, we choose $n \in \mathbb{N}$ such that $1/n < \delta$ and thus we have $|1/2n - 1/n| = 1/2n < \delta$ but $|f(1/n) - f(1/2n)| = n > 1 = \varepsilon$. Therefore, f is not uniformly continuous on $(0, 1]$.

Example 3.9 Let $0 < a < 1$. Define $f(x) = 1/x$ for $x \in [a, 1]$. Then f is uniformly continuous on $[a, 1]$. In fact for $x, y \in [a, 1]$, we have

$$|f(x) - f(y)| = |\frac{1}{x} - \frac{1}{y}| = \frac{|x-y|}{xy} \leq \frac{|x-y|}{a^2}.$$

So for any $\varepsilon > 0$, we can take $0 < \delta < a^2\varepsilon$. Thus if $x, y \in [a, 1]$ with $|x - y| < \delta$, then we have $|f(x) - f(y)| < \varepsilon$ and hence f is uniformly continuous on $[a, 1]$.

Proposition 3.10 *If f is continuous on a compact set A, then f is uniformly continuous on A.*

Proof: **Compactness argument**:
Let $\varepsilon > 0$. Since f is continuous on A, then for each $x \in A$, there is $\delta_x > 0$, such that $|f(y)-f(x)| < \varepsilon$ whenever $y \in A$ with $|y-x| < \delta_x$. Now for each $x \in A$, set $J_x = (x-\frac{\delta_x}{2}, x+\frac{\delta_x}{2})$. Then $A \subseteq \bigcup_{x\in A} J_x$. By the compactness of A, there are finitely many $x_1, ..., x_N \in A$ such that $A \subseteq J_{x_1} \cup \cdots \cup J_{x_N}$. Now take $0 < \delta < \min(\frac{\delta_{x_1}}{2}, ..., \frac{\delta_{x_N}}{2})$. Now for $x, y \in A$ with $|x - y| < \delta$, then $x \in I_{x_k}$ for some $k = 1, .., N$, from this it follows that $|x - x_k| < \frac{\delta_{x_k}}{2}$ and $|y-x_k| \leq |y-x|+|x-x_k| \leq \frac{\delta_{x_k}}{2} + \frac{\delta_{x_k}}{2} = \delta_{x_k}$. So for the choice of δ_{x_k}, we have $|f(y)-f(x_k)| < \varepsilon$ and $|f(x) - f(x_k)| < \varepsilon$. Thus we have shown that $|f(x) - f(y)| < 2\varepsilon$ whenever $x, y \in A$ with $|x - y| < \delta$. The proof is finished.
Sequentially compactness argument:
Suppose that f is not uniformly continuous on A. Then there is $\varepsilon > 0$ such that for each $n = 1, 2, ..$, we can find x_n and y_n in A with $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. Notice that by the sequentially compactness of A, (x_n) has a convergent subsequence (x_{n_k}) with $a := \lim_k x_{n_k} \in A$. Now applying sequentially compactness of A for the sequence (y_{n_k}), then (y_{n_k}) contains a convergent subsequence $(y_{n_{k_j}})$ such that $b := \lim_j y_{n_{k_j}} \in A$. On the other hand, we also have $\lim_j x_{n_{k_j}} = a$. Since $|x_{n_{k_j}} - y_{n_{k_j}}| < 1/n_{k_j}$ for all j, we see that $a = b$. This implies that $\lim_j f(x_{n_{k_j}}) = f(a) = f(b) = \lim_j f(y_{n_{k_j}})$. This leads to a contradiction since we always have $|f(x_{n_{k_j}}) - f(y_{n_{k_j}})| \geq \varepsilon > 0$ for all j by the choice of x_n and y_n above. The proof is finished. □

Proposition 3.11 *Let f be a continuous function defined on a bounded subset A of $\mathbb{R}$. Then the following statements are equivalent.*

- *(i): f is uniformly continuous on A.*
- *(ii): There is a unique continuous function F defined on the closure $\overline{A}$ such that $F(x) = f(x)$ for all $x \in A$.*

Proof: The Part $(ii) \Rightarrow (i)$ follows from Theorem 1.9 and Proposition 3.10 at once.
The proof of Part $(i) \Rightarrow (ii)$ is divided by the following assertions. Assume that f is uniformly continuous on A.
Claim 1. If (x_n) is a sequence in A and $\lim x_n$ exists, then $\lim f(x_n)$ exists.
It needs to show that $(f(x_n))$ is a Cauchy sequence. Indeed, let $\varepsilon > 0$. Then by the uniform continuity of f on A, there is $\delta > 0$ such that $|f(x)-f(y)| < \varepsilon$ whenever $x, y \in A$ with $|x-y| < \delta$. Notice that (x_n) is a Cauchy sequence since it is convergent. Thus, there is a positive integer

N such that $|x_m - x_n| < \delta$ for all $m, n \geq N$. This implies that $|f(x_m) - f(x_n)| < \varepsilon$ for all $m, n \geq N$ and hence, **Claim 1** follows.
Claim 2. If (x_n) and (y_n) both are convergent sequences in A and $\lim x_n = \lim y_n$, then $\lim f(x_n) = \lim f(y_n)$.
By **Claim 1**, $L := \lim f(x_n)$ and $L' = \lim f(y_n)$ both exist. For any $\varepsilon > 0$, let $\delta > 0$ be found as in **Claim 1**. Since $\lim x_n = \lim y_n$, there is $N \in \mathbb{N}$ such that $|x_n - y_n| < \delta$ for all $n \geq N$ and hence, we have $|f(x_n) - f(y_n)| < \varepsilon$ for all $n \geq N$. Taking $n \to \infty$, we see that $|L - L'| \leq \varepsilon$ for all $\varepsilon > 0$. So $L = L'$. **Claim 2** follows.
Recall that an element $x \in \overline{A}$ if and only if there is a sequence (x_n) in A converging to x.
Now for each $x \in \overline{A}$, we define

$$F(x) := \lim f(x_n)$$

if (x_n) is a sequence in A with $\lim x_n = x$. It follows from **Claim 1** and **Claim 2** that F is a well defined function defined on $\overline{A}$ and $F(x) = f(x)$ for all $x \in A$.
So, it remains to show that F is continuous. Then F is a continuous extension of f to $\overline{A}$ as deired.
Now suppose that F is not continuous at some point $z \in \overline{A}$. Then there is $\varepsilon > 0$ such that for any $\delta > 0$, there is $x \in \overline{A}$ satisfying $|x - z| < \delta$ but $|F(x) - F(z)| \geq \varepsilon$. Notice that for any $\delta > 0$ and if $|x - z| < \delta$ for some $x \in \overline{A}$, then we can choose a sequence (x_i) in A such that $\lim x_i = x$. Therefore, we have $|x_i - z| < \delta$ and $|f(x_i) - F(z)| \geq \varepsilon/2$ for any i large enough. Therefore, for any $\delta > 0$, we can find an element $x \in A$ with $|x - z| < \delta$ but $|f(x) - F(z)| \geq \varepsilon/2$. Now consider $\delta = 1/n$ for $n = 1, 2....$ This yields a sequence (x_n) in A which converges to z but $|f(x_n) - F(z)| \geq \varepsilon/2$ for all n. However, we have $\lim f(x_n) = F(z)$ by the definition of F which leads to a contradiction. Thus F is continuous on $\overline{A}$.
Finally the uniqueness of such continuous extension is clear.
The proof is finished. □

Example 3.12 By using Proposition 3.11, the function $f(x) := \sin \frac{1}{x}$ defined on $(0, 1]$ cannot be continuously extended to the set $[0, 1]$.

4 Lipschitz functions

Definition 4.1 Let A be a non-empty subset of $\mathbb{R}$. A function $f : A \to \mathbb{R}$ is called a Lipschitz if there is a constant $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in A$. In this case. Furthermore, if we can find such $0 < C < 1$, then we call f a contraction.

It is clear that we have the following property.

Proposition 4.2 *Every Lipschitz function is uniformly continuous on its domain.*

Example 4.3 (i) : The sine function $f(x) = \sin x$ is a Lipschitz function on $\mathbb{R}$ since we always have $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

(ii) : Define a function f on $[0, 1]$ by $f(x) = x \sin(1/x)$ for $x \in (0, 1]$ and $f(0) = 0$. Then f is continuous on $[0, 1]$ and thus f is uniformly continuous on $[0, 1]$. But notice that f is not a Lipschitz function. In fact, for any $C > 0$, if we consider $x_n = \frac{1}{2n\pi+(\pi/2)}$ and $y_n = \frac{1}{2n\pi}$, then $|f(x_n) - f(y_n)| > C|x_n - y_n|$ if and only if

$$\frac{2}{\pi} \cdot \frac{(2n\pi + \frac{\pi}{2})(2n\pi)}{2n\pi + \frac{\pi}{2}} = 4n > C.$$

Therefore, for any $C > 0$, there are $x, y \in [0, 1]$ such that $|f(x) - f(y)| > C|x - y|$ and hence f is not a Lipschitz function on $[0, 1]$.

Proposition 4.4 *Let A be a non-empty closed subset of $\mathbb{R}$. If $f : A \to A$ is a contraction, then there is a fixed point of f, that is, there is a point $a \in A$ such that $f(a) = a$.*

Proof: Since f is a contraction on A, there is $0 < C < 1$ such that $|f(x) - f(y)| \le C|x - y|$ for all $x, y \in A$. Fix $x_1 \in A$. Since $f(A) \subseteq A$, we can inductively define a sequence (x_n) in A by $x_{n+1} = f(x_n)$ for $n = 1, 2...$ Notice that we have

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \le C|x_n - x_{n-1}|$$

for all $n = 2, 3...$ This gives

$$|x_{n+1} - x_n| \le C^{n-1}|x_2 - x_1|$$

for $n = 2, 3, ...$ So, for any $n, p = 1, 2..$, we see that

$$|x_{n+p} - x_n| \le \sum_{i=n}^{n+p-1} |x_{i+1} - x_i| \le |x_2 - x_1| \sum_{i=n}^{n+p-1} C^{i-1}.$$

Since $0 < C < 1$, for any $\varepsilon > 0$, there is N such that $\sum_{i=n}^{n+p-1} C^{i-1} < \varepsilon$ for all $n \ge N$ and $p = 1, 2, ...$ Therefore, (x_n) is a Cauchy sequence and thus the limit $a := \lim_n x_n$ exists. Since A is closed, we have $a \in A$ and hence f is continuous at a. On the other hand, since $x_{n+1} = f(x_n)$. Therefore, we have $a = f(a)$ by taking $n \to \infty$. The proof is finished. □

Remark 4.5 The Proposition 4.4 does not hold if f is not a contraction. For example, if we consider $f(x) = x - 1$ for $x \in \mathbb{R}$, then it is clear that $|f(x) - f(y)| = |x - y|$ and f has no fixed point in $\mathbb{R}$.

5 Continuous functions defined on intervals

Recall that a non-empty subset I of $\mathbb{R}$ is called an interval if it has one of the following forms.

(i) $\mathbb{R}$.

(ii) $(-\infty, a]$ or $[a, \infty)$ or $(-\infty, a)$ or (a, ∞) for some $a \in \mathbb{R}$.

(iii) (a, b) or $(a, b]$ or $[a, b)$ or $[a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$.

Lemma 5.1 *Let I be a non-empty subset of $\mathbb{R}$. Suppose that there are different elements in I. Then I is an interval if and only if for any $a, b \in I$ with $a < b$, we have $[a, b] \subseteq I$.*

Proposition 5.2 (Intermediate Value Theorem): *Let I be an interval and let $f : I \to \mathbb{R}$ be a continuous function. Suppose that there are a and b in I with $f(a) < z < f(b)$. Then there is c between a and b such that $f(c) = z$.*

Proof: Notice that if we consider the function $x \in I \mapsto f(x) - z$, then we may assume that $z = 0$. Also, we may assume that $a < b$. Put $x_1 = a$ and $y_1 = b$. Now if $f(\frac{a+b}{2}) = 0$, then the result is obtained. If $f(\frac{a+b}{2}) > 0$, then we set $x_2 = a$ and $y_2 = \frac{a+b}{2}$. Similarly, if $f(\frac{a+b}{2}) < 0$, then we set $x_2 = \frac{a+b}{2}$ and $y_2 = b$. To repeat the same procedure, if there are x_N and y_N such that $f(\frac{x_N+y_N}{2}) = 0$, then the result is shown. Otherwise, we can find a decreasing sequence of closed and bounded intervals $[a, b] = [x_1, y_1] \supseteq [x_2, y_2] \supseteq \cdots$ with $\lim(y_n - x_n) = 0$ and $f(x_n) < 0 < f(y_n)$ for all n. Then by the Nested Intervals Theorem, we have $\bigcap_n [x_n, y_n] = \{c\}$ for some $c \in [x_1, y_1] = [a, b] \subseteq I$ because I is an interval. Moreover, we have $\lim_n x_n = \lim_n y_n = c$. Then by the continuity of f, we see that $f(c) = \lim f(x_n) = \lim f(y_n)$. Since $f(x_n) < 0 < f(y_n)$ for all n, we have $f(c) = 0$. The proof is finished. $\square$

Remark 5.3 The assumption of the intervals in the Intermediate Value Theorem is essential. For example, consider $I = [0, 1) \cup (2, 3]$ and define $f : I \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in [0,1) \\ x - 1 & \text{if } x \in (2,3]. \end{cases}$$

Then $f(0) < 1 < f(3)$ but $1 \notin f(I)$.

Corollary 5.4 *Let $f; [a, b] \to \mathbb{R}$. Suppose that $M := \sup\{f(x) : x \in [a, b]\}$ and $m = \inf\{f(x) : x \in [a, b]\}$. Then $f([a, b]) = [m, M]$.*

Proof: Notice that if $m = M$, then f is a constant function and hence, the result is clearly true.
Now suppose that $m < M$. It is clear that $f([a, b]) \subseteq [m, M]$ because $m \leq f(x) \leq M$ for all $x \in [a, b]$. For the converse inclusion, notice that since $[a, b]$ is compact, there are x_1 and x_2 in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$. We may assume that $x_1 < x_2$. To apply the Intermediate Value Theorem for the restriction of f on $[x_1, x_2]$, we have $[m, M] \subseteq f([x_1, x_2]) \subseteq f([a, b])$. The proof is finished. $\square$

Corollary 5.5 *Let I be an interval and let $f : I \to \mathbb{R}$ be a continuous non-constant function. Then $f(I)$ is an interval.*

Proof: Notice that by Lemma 5.1, it needs to show that for any $c, d \in f(I)$ with $c < d$ implies that $[c, d] \subseteq f(I)$. Suppose that $a, b \in I$ with $a < b$ satisfy $f(a) = c$ and $f(b) = d$. Notice that $[a, b] \subseteq I$ because I is an interval. If we put $M = \sup_{x\in[a,b]} f(x)$ and $m = \inf_{x\in[a,b]} f(x)$, then by Corollary 5.4, we have

$$[c, d] \subseteq [m, M] = f([a, b]) \subseteq f(I).$$

The proof is finished. $\square$

Example 5.6 It is impossible to find a continuous surjection from (a, b) onto $(c, d) \cup (e, f)$ where $d \leq e$.

References

[1] R.G. Bartle and I.D. Sherbert, Introduction to Real Analysis, (4*th ed*), Wiley, (2011).